Section 3.3: Linear programming: A geometric approach

In addition to constraints, linear programming problems usually involve some quantity to maximize or minimize such as profits or costs. The quantity to be maximized or minimized translates to some linear combinations of the variables called an objective function. These problems involve choosing a solution from the feasible set for the constraints which gives an optimum value (maximum or a minimum) for the objective function.

Example A juice stand sells two types of fresh juice in 12 oz cups. The Refresher and the Super Duper. The Refresher is made from 3 oranges, 2 apples and a slice of ginger. The Super Duper is made from one slice of watermelon and 3 apples and one orange. The owners of the juice stand have 50 oranges, 40 apples, 10 slices of watermelon and 15 slices of ginger. Let x denote the number of Refreshers they make and let y denote the number of Super Dupers they make.

Last day, we saw that the **set of constraints** on x and y were of the form :

$$3x + y \leq 50$$

$$2x + 3y \leq 40$$

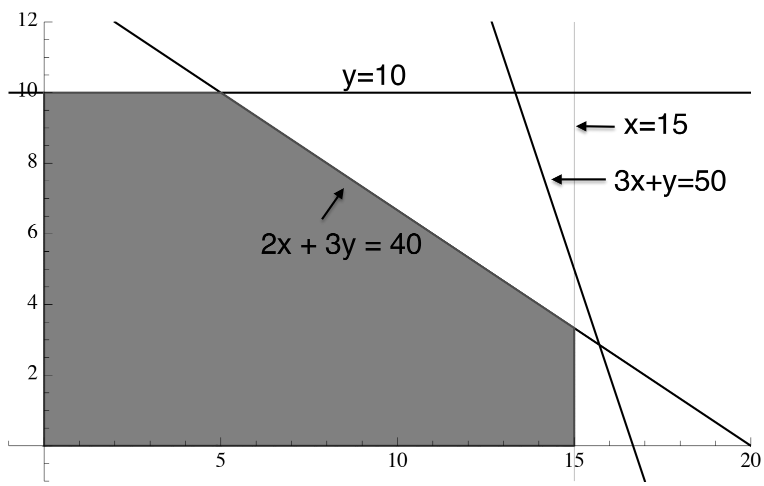
$$x \leq 15$$

$$y \leq 10$$

$$x \geq 0$$

$$y \geq 0$$

The conditions $x \geq 0$ and $y \geq 0$ are called **non-negative conditions.** We can now draw the feasible set to see what combinations of x and y are possible given the limited supply of ingredients:



Now suppose that the Refreshers sell for \$6 each and the Super Dupers sell for \$8 each. Lets suppose also that the juice stand will sell all of the drinks they can make on this day, then their revenue for the day is $6x + 8y$. Lets assume also that one of the goals of the juice stand is to maximize revenue. Thus they want to maximize the value of $6x + 8y$ given the constraints on production listed above. In other words they want to find a point (x, y) in the feasible set which gives a maximum value for the **objective function** $6x + 8y$. [Note that the value of the objective function ($6x + 8y$ = revenue) varies as (x, y) varies over the points in the feasible set. For example if $(x, y) = (2, 5)$, revenue $= 6(2) + 8(5) = \$52$, whereas if $(x, y) = (5, 10)$, revenue $= 6(5) + 8(10) = \$110$.]

Terminology: A linear inequality of the form

$$a_1x + a_2y \leq b, \quad a_1x + a_2y < b, \quad a_1x + a_2y \geq b, \quad \text{or} \quad a_1x + a_2y > b,$$

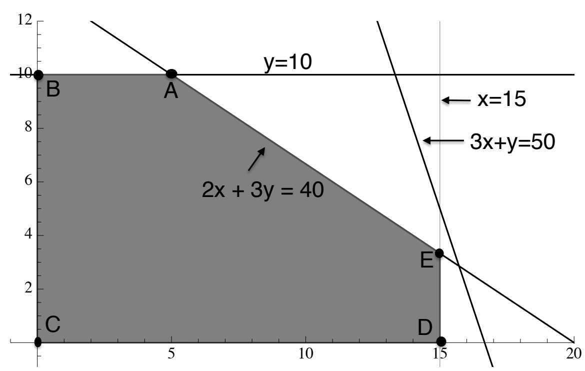
where a_1, a_2 and b are constants, is called a **constraint** in a linear programming problem. The restrictions $x \geq 0$, $y \geq 0$ are called **nonnegative conditions**. A **linear objective function** is an expression of the form $cx + dy$, where c and d are constants, for which one needs to find a maximum or minimum on the feasible set. The term **optimal value** refers to the sought after maximum or minimum as the case may be.

Theorem Given a linear objective function subject to constraints in the form of linear inequalities, if the objective function has an optimal value (maximum or minimum) on the feasible set, it must occur at a corner(or vertex) of the feasible set.

Example from above We can summarize our problem from the previous example in the following form:
Find the maximum of the objective function $6x + 8y$ subject to the constraints

$$\begin{array}{cc} 3x + y \leq 50, & 2x + 3y \leq 40, \\ x \leq 15, & y \leq 10, \\ x \geq 0, & y \geq 0. \end{array}$$

From the above theorem, we know that the maximum of $6x + 8y$ on the feasible set occurs at a corner of the feasible set (it may occur at more than one corner, but it occurs at at least one). We already have a picture of the feasible set and below, we have labelled the corners, A, B, C, D and E.



To find the maximum value of $6x + 8y$ and a point (x, y) in the feasible set at which it is achieved, we need only calculate the co-ordinates of the points A, B, C, D and E and compare the value of $6x + 8y$ at each.

Point	Coordinates	Value of $6x + 8y$
A		
B		
C	(0, 0)	0
D		
E		

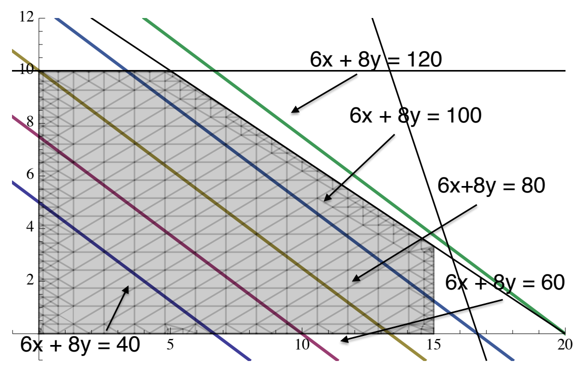
Point	Coordinates	Value of $6x + 8y$
A	(5,10)	110
B	(0,10)	80
C	(0, 0)	0
D	(15,0)	90
E	(15,10/3)	$\frac{350}{3}$

Hence E is the largest value (and C is the smallest).

Method for finding optimal values: Given a set of linear constraints and a linear objective function, to find the optimal value of the objective function subject to the given constraints;

- Draw the feasible set and identify which lines form the boundaries.
- Calculate the coordinates of the corners by finding the intersection of the relevant lines.
- Evaluate the objective function at each corner.
- Determine which corners give the optimal value (maximum or minimum) of the objective function.

To get some idea of why the optimal values occur at the corners, look at the picture below. It shows lines of the form $6x + 8y = c$ where c takes the values $40, 60, 80, 100$ and 120. The value of the objective function increases along a line perpendicular to those drawn, so the maximum value of the objective function will appear somewhere along the boundary of the feasible set. The set of points at which the maximum is achieved will always include a vertex (in our case it is limited to a vertex) but may include a portion of a line along the boundary.



Given the assumptions of the above theorem:

- **If the feasible set is bounded** and if all lines on the boundary of the feasible set are in the feasible set, a maximum and a minimum value of the objective function will exist among the values at the corner points.
- **If the feasible set is unbounded** optimal values may or may not exist. A more detailed discussion follows at the end of these notes.
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  - **If the feasible set is unbounded** and
  - if the constraints have non-negative conditions and
  - if the objective function is of the form $cx + dy$ where c and d are > 0

  then the objective function will achieve a minimum value at a corner of the unbounded feasible set, but will have no maximum.

Example Mr. Carter eats a mix of Cereal A and Cereal B for breakfast. The amount of calories, sodium and protein per ounce for each is shown in the table below. Mr. Carter's breakfast should provide at least 480 calories but less than or equal to 700 milligrams of sodium. Mr. Carter would like to maximize the amount of protein in his breakfast mix.

	Cereal A	**Cereal B**
Calories(per ounce)	100	140
Sodium(mg per ounce)	150	190
Protein(g per ounce)	9	10

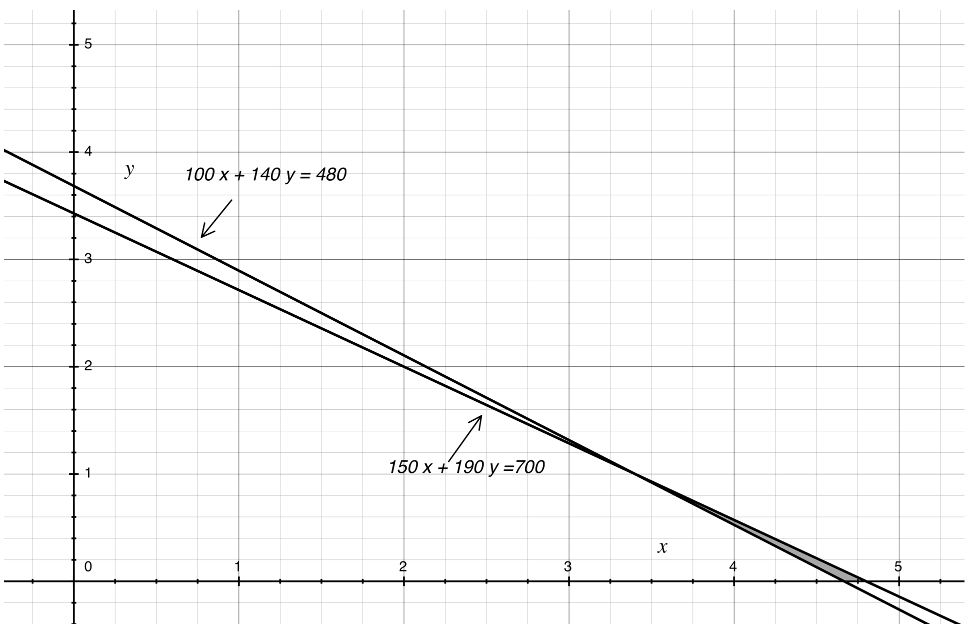
Let x denote the number of ounces of Cereal A that Mr. Carter has for breakfast and let y denote the number of ounces of Cereal B that Mr. Carter has for breakfast. In the last lecture, we found that the set of constraints for x and y were

$$100x + 140y \geq 480, \qquad 150x + 190y \leq 700, \qquad x \geq 0, \quad y \geq 0$$

(a) What is the objective function?

$9x + 10y$.

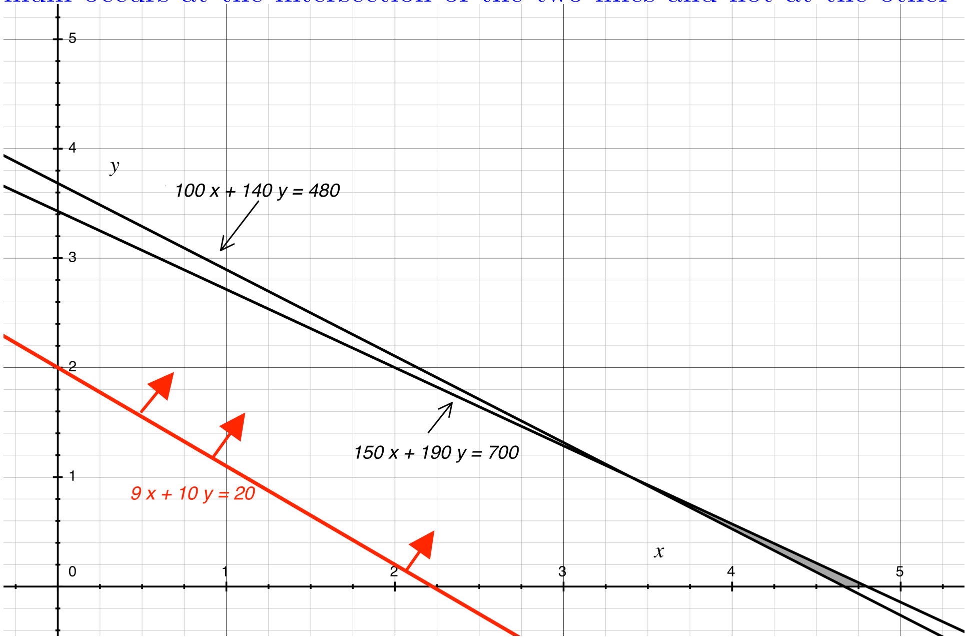
(b) Graph the feasible set. (Note: Once can multiply or divide an inequality by a positive number without changing the solution set just as one would an equation. [When we multiply or divide an inequality by a negative number you must reverse the inequality])



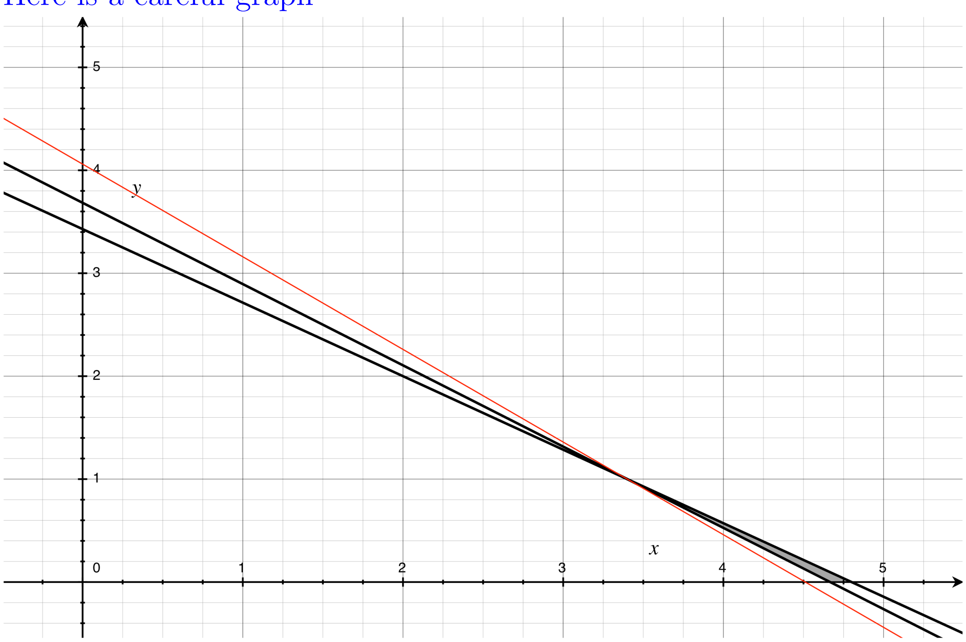
(c) Find the vertices of the feasible set and the maximum of the objective function on the feasible set.

The two vertices on the x-axis are $(4.8, 0)$ and $\left(\frac{16}{3}, 0\right)$. The intersection of $100x+140y = 480$ and $150x + 190y = 700$ is the point $\left(\frac{17}{5}, 1\right)$. The values of the objective function are 42.2, 48 and $\frac{203}{5} = 40.6$. Hence the maximum of the objective function is 48 and it occurs at the point $\left(\frac{16}{3}, 0\right)$ on the boundary of the feasible region and nowhere else. The minimum occurs at 40.6 where the two lines intersect.

In the next graph we have included a red line which is the objective equation with value 20. The red arrows indicate which way to move to increase the value of the objective function. From the picture it seems clear that the maximum value occurs at the vertex on the x–axis as far to the right as possible. It is harder to see that the minimum occurs at the intersection of the two lines and not at the other vertex on the x–axis.



Here is a careful graph

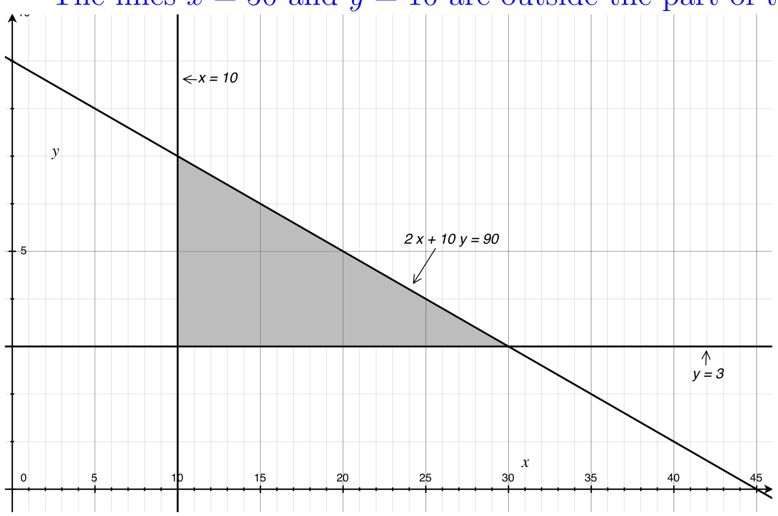


This porblem is a slight change in the feasible set from the previous version where Mr. Carter wanted less than 700 milligrams of sodium. In terms of the picture of the feasible set, the line $150x+190y = 700$ was not in the feasible set in the previous lecture but it is here. Linear programing problems where some of the lines are not in the feasible set are tricky. All you can do is first pretend that all the lines **are** in the feasible set, do the problem and if the solution is at a corner where either (or both) lines are dotted, use your common sense.

Example Michael is taking a timed exam in order to become a volunteer firefighter. The exam has 10 essay questions and 50 multiple choice questions. He has 90 minutes to take the exam and knows he cannot possibly answer every question. The essay questions are worth 20 points each and the short-answer questions are worth 5 points each. An essay question takes 10 minutes to answer and a shot-answer question takes 2 minutes. Michael must do at least 3 essay questions and at least 10 short-answer questions. Michael knows the material well enough to get full points on all questions he attempts and wants to maximize the number of points he will get. Let x denote the number of multiple choice questions that Michael will attempt and let y denote the number of essay questions that Michael will attempt. Write down the constraints and objective function in terms of x and y and find the/a combination of x and y which will allow Michael to gain the maximum number of points possible.

$2x + 10y \leqslant 90$ (time needed to answer the questions).
$x \geqslant 10$ (at least 10 short-answer questions).
$x \leqslant 50$ (at most 50 short-answer questions).
$y \geqslant 3$ (at least 3 essay questions).
$y \leqslant 10$ (at least 3 essay questions).
$5x + 20y$ is the objective function (Michael's total score).
Here is the feasible region.

The lines $x = 50$ and $y = 10$ are outside the part of the plane shown.



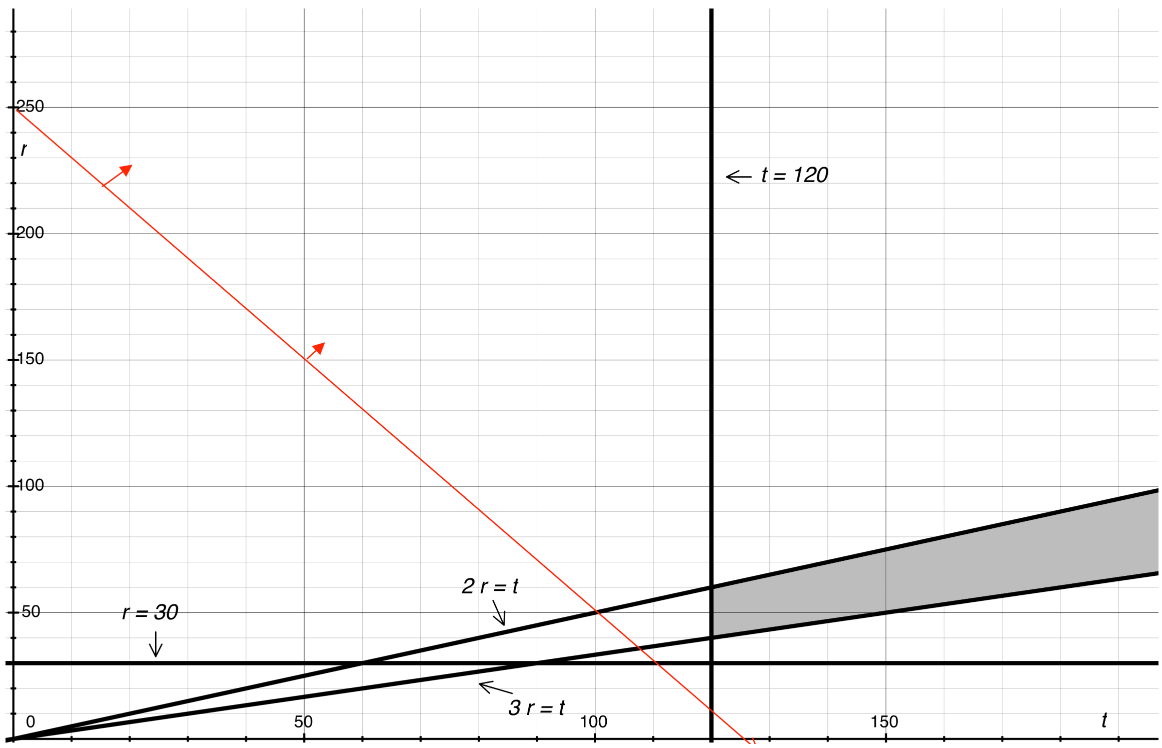
The three vertices are $(10, 3)$, $(10, 7)$ and $(30, 3)$. The values of the objective function are 60, 190 and 210. Hence Michael can maximize his score by answering 3 essay questions and 30 short-answer questions.

Example with unbounded region A local politician is budgeting for her media campaign. She will distribute her funds between TV ads and radio ads. She has been given the following advice by her campaign advisors;

- She should run at least 120 TV ads and at least 30 radio ads.
- The number of TV ads she runs should be at least twice the number of radio ads she runs but not more than three times the number of radio ads she runs.

The cost of a TV ad is \$8000 and the cost of a radio ad is \$2000. Which combination of TV and radio ads should she choose to minimize the cost of her media campaign?

Let t be the number of TV ads and r the number of radio ads. $t \geqslant 120$ and $r \geqslant 30$.
$2r \leqslant t \leqslant 3r$.
The objective function is $8000t + 2000r$ which she wishes to minimize.
Below is the feasible region which we see is unbounded. The red line is a value of the objective function and the red arrow indicate the direction you should move to increase its value. Hence there is a minimum where $t = 120$ intersects $3r = t$ or $t = 120$, $r = 40$ at a cost of $1,040,000$.



She can always increase her cost by buying more ads so there is no maximum.

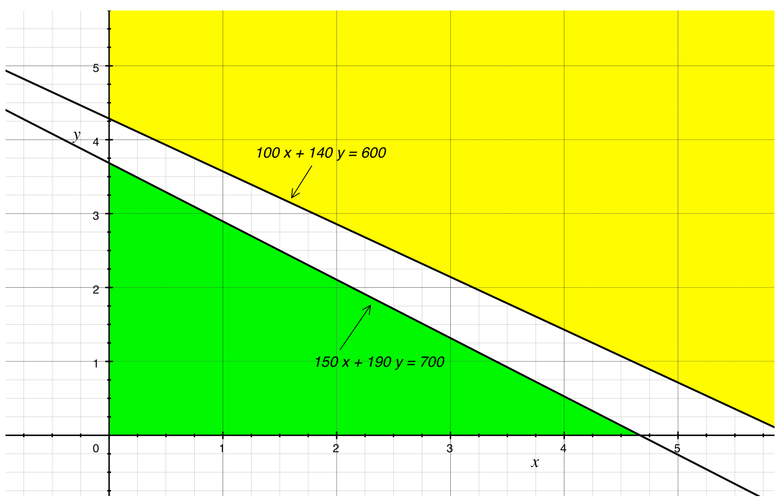
Example: No feasible region Mr. Baker eats a mix of Cereal A and Cereal B for breakfast. The amount of calories, sodium and protein per ounce for each is shown in the table below. Mr. Baker's breakfast should provide at least 600 calories but less than 700 milligrams of sodium. Mr. Baker would like to maximize the amount of protein in his breakfast mix.

	Cereal A	**Cereal B**
Calories(per ounce)	100	140
Sodium(mg per ounce)	150	190
Protein(g per ounce)	9	10

Let x denote the number of ounces of Cereal A that Mr. Baker has for breakfast and let y denote the number of ounces of Cereal B that Mr. Baker has for breakfast.

(a) Show that the feasible set is empty here, so that there is no feasible combination of the cereals for Mr. Baker's breakfast.

$100x + 140y \geqslant 600$ (calories)
$150x + 190y < 700$ (sodium)
$x \geqslant 0$, $y \geqslant 0$



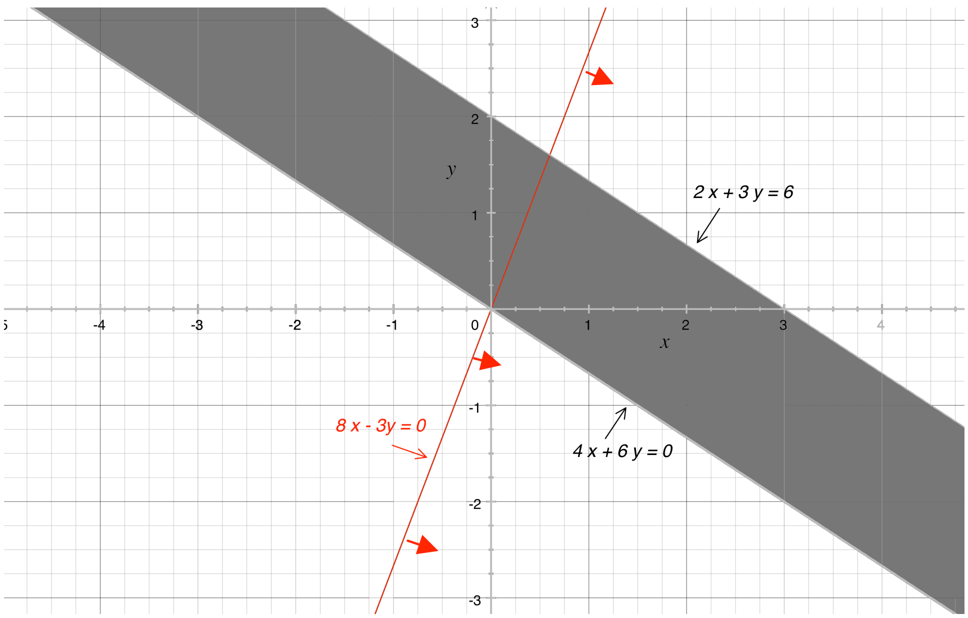
The yellow region is $100x + 140y \geqslant 600$ and the green region without the line is $150x + 190y < 700$ and clearly the two regions have no intersection.

(b) If Mr. Baker goes shopping for new cereals, what should he look for on the chart giving the Nutritional value, so that he can have some feasible combination of the cereals for breakfast?

This is an essay question with no single right answer. Basically Mr. Baker needs to choose cereals with either more calories or less sodium per ounce.

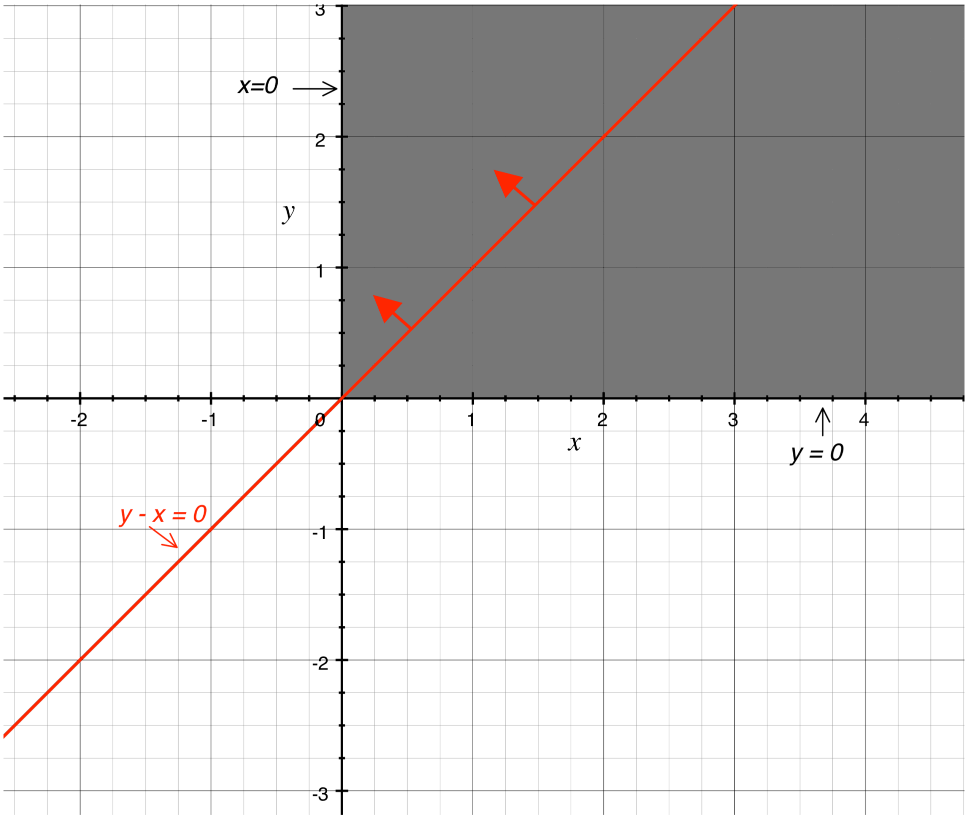
Example: Unbounded feasible region with no optimal values Consider constraints $2x + 3y \leqslant 6$

and $4x + 6y \geqslant 0$ and objective function $8x - 3y$. Here is the picture.



Much about this picture is true in general. If the region is unbounded and has no vertices then it is the region between two parallel lines. If the objective function is not parallel to the boundary lines then it has neither a minimum nor a maximum. If it is parallel to the boundary, then it has both a minimum and a maximum. In the example above, the objective function $-x - 1.5y$ has a maximum of 0 along the lower line and a minimum of -3 along the upper line.

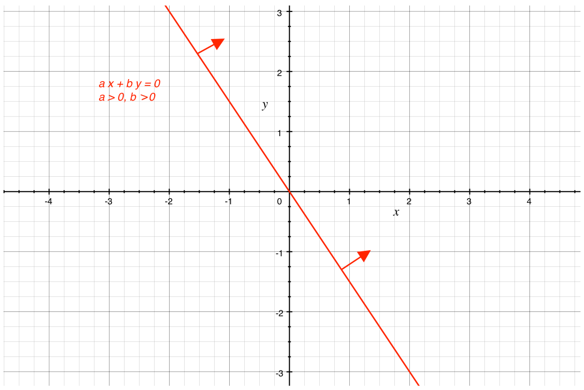
Consider constraints $x \geqslant 0$ and $y \geqslant 0$ and objective function $y - x$. Here is the picture.

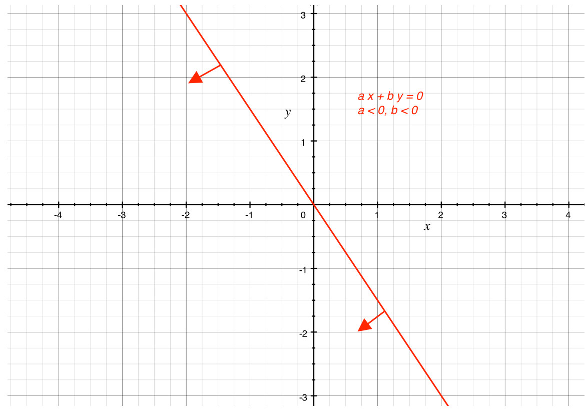


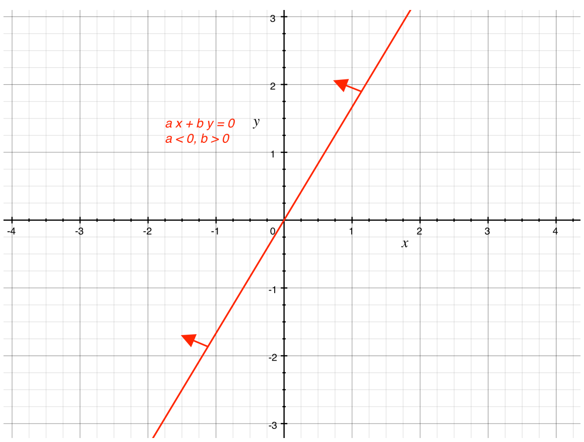
Notice you have a vertex but no maximum or minimum.

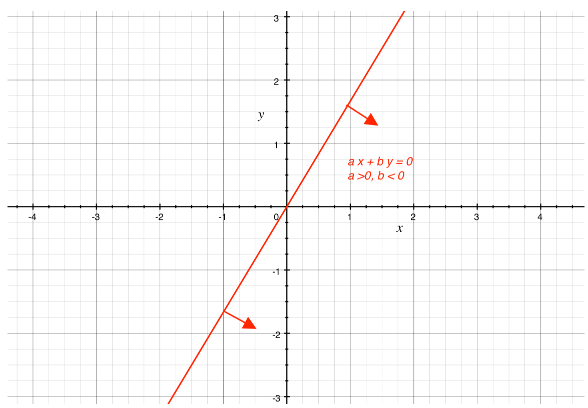
The complete discussion of optimal values

Any linear objective function defines a family of parallel lines. Each of them divides the plane into two pieces, one side of the line is the direction in which the function is increasing. Here are four examples with the increasing direction pointed to by the red arrows.



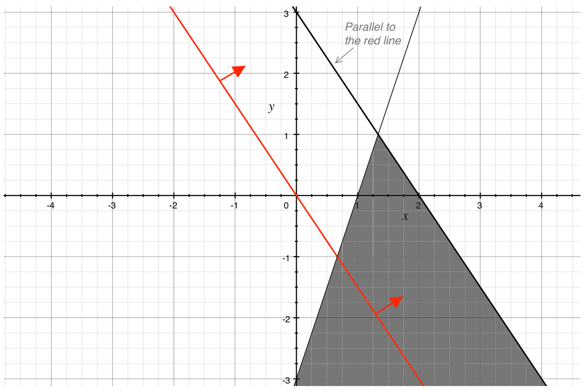


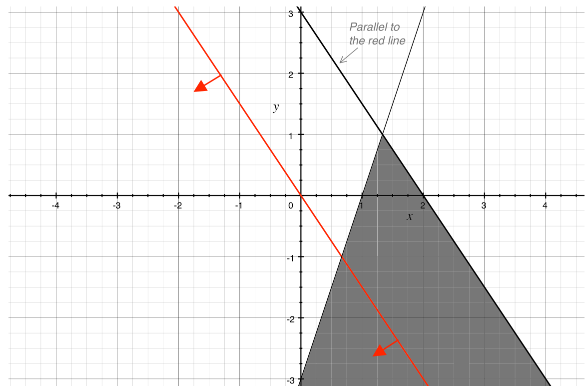




The existence of a maximal optimal value or a negative optimal value depends on how the objective function line intersects the boundary of the feasible set.

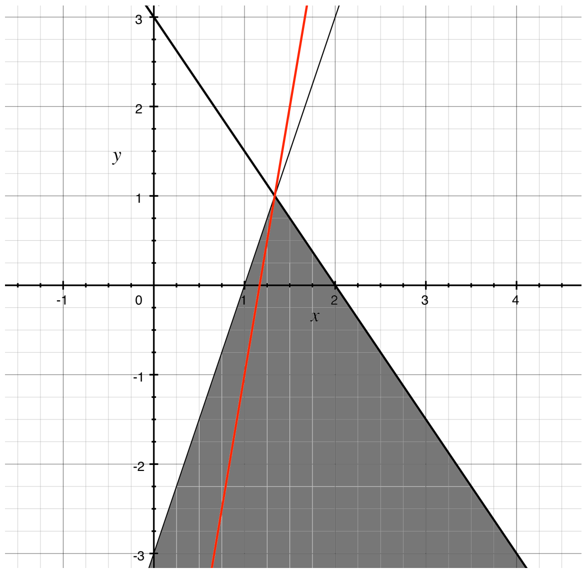
Case 1: The objective function line is parallel to one of the boundary lines of the feasible set. Then the value of the object function on that boundary line is either a maximum or a minimum. If the red arrows point into the feasible set then the value is a minimum. If the red arrows point out of the feasible set then the value is a maximum.

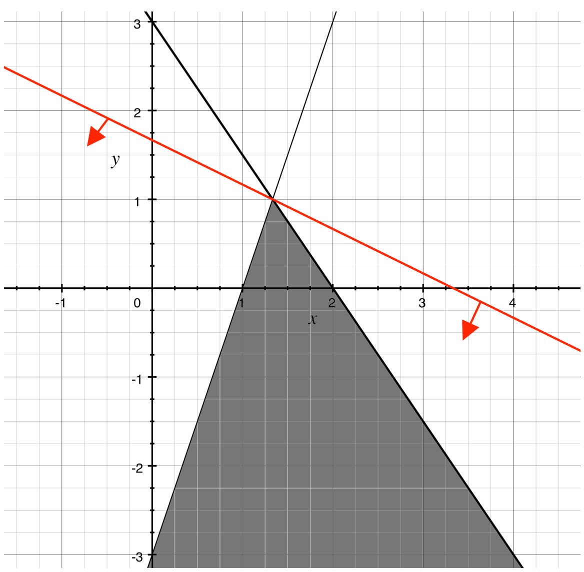


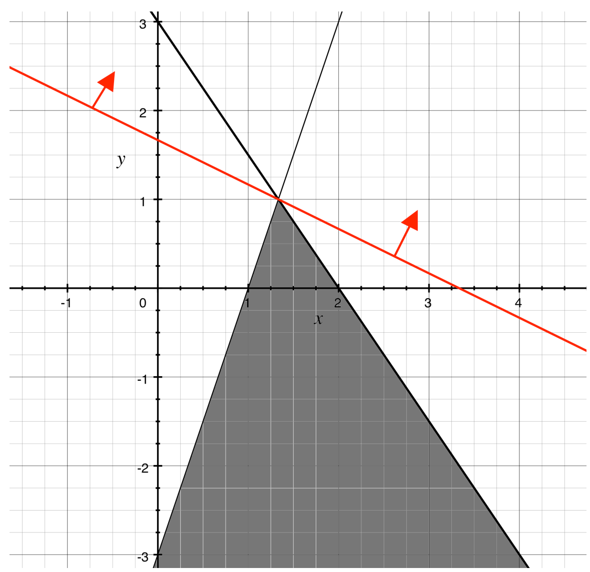


In this example, the red objective function line is parallel to the indicated boundary of the feasible set. In the left hand picture, the indicated boundary line is a minimum while the right hand picture, the indicated boundary line is a maximum.

Case 2: The objective function line is not parallel to any of the boundary lines of the feasible set. Then any optimal value must occur at a vertex. There are two possibilities.







The upper picture occurs whenever the vertex is neither a maximum nor a minimum. One of the two lower pictures occurs whenever the vertex is an optimal value. The left hand lower picture is a minimum; the right hand lower picture is a maximum.

Here is the first example we worked with the objective function $6x + 8y$. The objective line is not parallel to any of the boundary lines so we only need to look at the lines going through the five vertices.